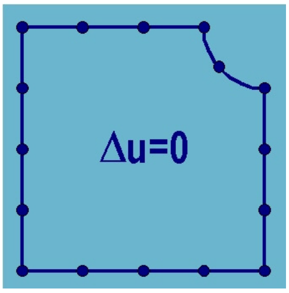

Fundamental solution of the 3D Laplace equation

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Fundamental solution of the 3D Laplace equation

In this case a transformation on spherical coordinates is carried out

$$G_{,ii} = \frac{1}{r^2}\left[\left(r^2 G_{,r}\right)_{,r} + \frac{1}{\sin^2\vartheta} G_{,\varphi\varphi} + G_{,\vartheta\vartheta} + \frac{1}{\tan\vartheta} G_{,\vartheta}\right] . \tag{162}$$

Assuming radial symmetry yields

$$G_{,ii} = \frac{1}{r^2}\left(r^2 G_{,r}\right)_{,r} . \tag{163}$$

The Dirac impulse in spherical coordinates is

$$\delta\left(x_1\right)\delta\left(x_2\right)\delta\left(x_3\right) = \frac{\delta\left(r\right)}{4\pi r^2} \tag{164}$$

since

$$\int\limits_{\Omega_\infty} \delta\left(x_1\right)\delta\left(x_2\right)\delta(x_3)\; d\Omega = \int\limits_{-\infty}^{\infty} \delta(x_1)\; dx_1 \int\limits_{-\infty}^{\infty} \delta(x_2)\; dx_2 \int\limits_{-\infty}^{\infty} \delta(x_3)\; dx_3 = 1 \tag{165}$$

and by equivalence

$$\int\limits_{\Omega_\infty} \frac{\delta\left(r\right)}{4\pi r^2}\, d\Omega \quad = \int\limits_0^\infty \frac{\delta\left(r\right)}{4\pi r^2}\, r^2\, dr \int\limits_0^\pi \sin\vartheta\, d\vartheta \int\limits_0^{2\pi} d\varphi = 1\,. \tag{166}$$

The integration of the Laplace equation yields for the 3D case

$$\begin{aligned} \frac{1}{r^2}\left(r^2 G_{,r}\right)_{,r} &= -\frac{\delta\left(r\right)}{4\pi r^2} \\ r^2 G_{,r} &= -\int \frac{\delta\left(r\right)}{4\pi}\, dr = -\frac{1}{4\pi} - C_1 \\ G_{,r} &= -\frac{1}{4\pi r^2} - \frac{C_1}{r^2} \\ G(r) &= \frac{1}{4\pi r} + \frac{C_1}{r} + C_2\,. \end{aligned} \tag{167}$$

Again, the impulse condition will show that $C_1 = 0$. As before C_2 is arbitrary and is set to zero for convenience.

Verification of the impulse condition

The derivation is analogous to the 2D case. The integral over the partial differential equation is transformed to the boundary. Since the solution is radial symmetric the gradient has only a component in the radial direction.

$$G_{,i} = \left[\begin{array}{ccc} G_{,r} & 0 & 0 \end{array}\right]^{\mathrm{T}} \tag{168}$$

A sphere is chosen as arbitrary enclosing surface in the 3D case. The normal vector is a unit vector in spherical coordinates

$$\boldsymbol{n} = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]^{\mathrm{T}}. \tag{169}$$

With this the impulse condition is

$$\int_\Gamma G_{,r} \mathrm{d}\Gamma = -1 \; . \tag{170}$$

Again, G only depends on r and thus is constant on a sphere of constant radius. It follows

$$\begin{aligned} G_{,r} \int_\Gamma d\Gamma &= -1 \\ G_{,r}\Gamma &= -1 \\ -\frac{1}{4\pi r^2} 4\pi r^2 &= -1 \; . \end{aligned} \tag{171}$$

As in the 2D case, this shows that C_1 must be set to zero so that G fulfills this equation.

The solutions for the potential G and the flux G_n as the normal derivative of the potential in 2D and 3D are summarized in Table 2.

Table 2: Fundamental solutions of the Laplace equation

	2D	3D
$G(r)$	$-\frac{1}{2\pi}\ln r$	$\frac{1}{4\pi r}$
$G_n(r) = \frac{\partial G}{\partial n}$	$-\frac{1}{2\pi r}\frac{\partial r}{\partial n}$	$-\frac{1}{4\pi r^2}\frac{\partial r}{\partial n}$

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